PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 5, Fasc. 1 (1985), p. 1-20

SCALE INVARIANCE OF STATISTICAL EXPERIMENTS

BY

HELMUT STRASSER (BAYREUTH)

Abstract. It has been shown by Le Cam [5] that weak limits of experiments, which are parametrized in a certain way, typically satisfy an invariance condition which is called translation invariance. It is shown in the present paper that weak limits of product experiments with identical factors can be characterized by stability. This property has been considered already by Müller [10] under the label scale invariance. There is given a complete description of all Gaussian experiments which are translation and stable. Any translation invariant experiment with finite-dimensional parameter space which is stable with exponent $p = 2$ must be a Gaussian shift. These results specify and extend indications of Müller [10].

1. INTRODUCTION

Let $T \neq \emptyset$ be an arbitrary set. A statistical experiment $E = (\Omega, \mathscr{A}, \mathscr{P})$ for the parameter space T is a triplet consisting of a measurable space $(\Omega, \mathscr{A})$ and a family $\mathscr{P} = \{P_t: t \in T\}$ of probability measures. The collection of all experiments for the parameter space T is denoted by $\mathscr{E}(T)$. The experiments for which $t \mapsto P_t$ is constant are called *trivial.*

From basic results of decision theory an equivalence relation on $\mathscr{E}(T)$ arises. The set of equivalence classes is denoted by $\dot{\mathscr{E}}(T)$. In the sequel the weak topology on $\dot{\mathscr{E}}(T)$ is considered. For both, the equivalence relation and the weak topology, the reader is referred to the literature, e.g. Le Cam [6] and Strasser [14].

Let $A(T)$ be the family of all finite subsets of T. If $\alpha \in A(T)$ and $E_\alpha = (\Omega, \mathscr{A}, \mathscr{P}_\alpha)$ with $\mathscr{P}_\alpha = \{P_t: t \in \alpha\}$, then the *Hellinger transform* of E_α is the function

$$H(E_\alpha):\ z \mapsto \int \prod_{t \in \alpha} \left(\frac{dP_t}{dv}\right)^{z_t} dv, \qquad z \in S_\alpha,$$

where

$$S_\alpha = \{z \in \boldsymbol{R}^\alpha \colon 0 \leqslant z_t \leqslant 1,\ t \in \alpha,\ \sum_{t\in\alpha} z_t = 1\}$$

and $\nu|\mathscr{A}$ is an arbitrary σ-finite measure dominating $\mathscr{P}_\alpha$.

It is well-known that two experiments E and F in $\mathscr{E}(T)$ are equivalent iff

$$H(E_\alpha) = H(F_\alpha) \quad \text{for every } \alpha \in A(T).$$

Moreover, a sequence $(E_n)_{n\in N}$ converges weakly to E iff for every $\alpha \in A(T)$

$$\lim_{n\to\infty} H(E_{n,\alpha}) = H(E_\alpha) \quad \text{on } S_\alpha.$$

Let us illustrate the weak topology by some typical examples.

(1.1) Example. For every $p \in [0, 1]$ let $B(p) = (1-p)\varepsilon_0 + p\varepsilon_1$ be a two-point measure on $(\boldsymbol{R}, \mathscr{B})$. Fix some $p \in (0, 1)$. The sets

$$T_n := \{t \in \boldsymbol{R} \colon p + tn^{-1/2} \in [0, 1]\}$$

satisfy $T_n \uparrow T := \boldsymbol{R}$. Then the sequence of experiments

$$E_n := (\boldsymbol{R}^n, \mathscr{B}^n, \{B^n(p+tn^{-1/2}) \colon t \in T_n\}), \qquad n \in \boldsymbol{N},$$

converges weakly to the Gaussian shift experiment $F = (\boldsymbol{R}, \mathscr{B}, \{\nu_{t,\sigma^2} \colon t \in \boldsymbol{R}\})$ with $\sigma^2 = p(1-p)$.

(1.2) Example. For every $\lambda \geqslant 0$ let $P(\lambda)$ be the Poisson distribution with mean λ. The sets $T_n := \{\lambda \geqslant 0 \colon \lambda n^{-1} \in [0, 1]\}$ satisfy $T_n \uparrow T := [0, \infty)$. Then the sequence of experiments

$$E_n := (\boldsymbol{R}^n, \mathscr{B}^n, \{B^n(\lambda n^{-1}) \colon \lambda \in T_n\}), \qquad n \in \boldsymbol{N},$$

converges weakly to the experiment $F = (\boldsymbol{R}, \mathscr{B}, \{P(\lambda) \colon \lambda \geqslant 0\})$.

(1.3) Example. Denote the unit vectors of $\boldsymbol{R}^m$ by e_i, $1 \leqslant i \leqslant m$, and define the simplex

$$S_m = \{p \in \boldsymbol{R}^m \colon p_i > 0,\ 1 \leqslant i \leqslant m,\ \sum_{i=1}^m p_i = 1\}.$$

For every $p \in S_m$ let

$$M(p) = \sum_{i=1}^m p_i \varepsilon_{e_i}.$$

Define a hyperplane

$$T := \{t \in \boldsymbol{R}^m \colon \sum_{i=1}^m t_i = 0\}.$$

Fix some $p \in S_m$. The sets

$$T_n := \{t \in T \colon p + tn^{-1/2} \in S_m\}$$

satisfy $T_n \uparrow T$. Then the sequence of experiments

$$E := (\mathbf{R}^{mn}, \mathscr{B}^{mn}, \{M^n(p+tn^{-1/2}): t \in T_n\}), \qquad n \in \mathbf{N},$$

converges weakly to the Gaussian shift experiment

$$F = (\mathbf{R}^m, \mathscr{B}^m, \{\nu_{t,\Gamma}: t \in T\})$$

with $\Gamma = (p_i \delta_{ij})_{1 \leqslant i,j \leqslant m}$.

(1.4) Example. Consider now the family of probability measures $\{P_\vartheta | \mathscr{B}: \vartheta \in \mathbf{R}\}$ with Lebesgue densities

$$\frac{dP_\vartheta}{d\lambda}(x) = C(\alpha) \exp(-|x-\vartheta|^\alpha), \qquad x \in \mathbf{R},$$

where $\alpha > 0$ is a characteristic constant of the family. Let

$$\delta_n = \begin{cases} n^{-1/2} & \text{if } \alpha > 1/2, \\ (n \log n)^{-1/2} & \text{if } \alpha = 1/2, \\ n^{-1/(2\alpha+1)} & \text{if } \alpha < 1/2. \end{cases}$$

For some fixed $\vartheta \in \mathbf{R}$ consider the sequence of experiments

$$E_n := (\mathbf{R}^n, \mathscr{B}^n, \{P^n_{\vartheta+\delta_n t}: t \in \mathbf{R}\}), \qquad n \in \mathbf{N}.$$

If $\alpha \geqslant 1/2$, then $(E_n)_{n \in \mathbf{N}}$ converges weakly to the experiment

$$F = (\mathbf{R}, \mathscr{B}, \{\nu_{t,\sigma^2}: t \in \mathbf{R}\}),$$

where $\sigma^2 > 0$ depends on α. If $\alpha < 1/2$, then $(E_n)_{n \in \mathbf{N}}$ converges weakly to an experiment $F = (\Omega, \mathscr{A}, \{Q_t: t \in \mathbf{R}\})$ which is characterized by the fact that

$$\left(\log \frac{dQ_t}{dQ_0}\right)_{t \in \mathbf{R}}$$

is a Gaussian process with covariance

$$K(s, t) = c(\alpha)(|s|^{2\alpha+1} + |t|^{2\alpha+1} - |s-t|^{2\alpha+1})$$

and mean $-K(t, t)/2$, $s \in \mathbf{R}$, $t \in \mathbf{R}$.

(1.5) Example. Let $P_0 | \mathscr{B}$ be the probability measure with Lebesgue density

$$\frac{dP_0}{d\lambda} = \tfrac{1}{2} \mathbf{1}_{(-1,+1)}.$$

Consider the family of probability measures $P_\vartheta = P_0 * \varepsilon_\vartheta$, $\vartheta \in \mathbf{R}$. Then, for some fixed $\vartheta \in \mathbf{R}$, the sequence of experiments

$$E_n = (\mathbf{R}^n, \mathscr{B}^n, \{P^n_{\vartheta+tn^{-1}}: t \in \mathbf{R}\}), \qquad n \in \mathbf{N},$$

converges weakly to the experiment $F = (R^2, \mathscr{B}^2, \{Q_t\colon t \in R\})$, where

$$\frac{dQ_t}{d\lambda^2}(x, y) = e^{x-y} 1_{(-\infty,t)\times(t,\infty)}(x, y), \qquad (x, y) \in R^2,\ t \in R.$$

The common feature of the preceding examples is that they are dealing with weak convergence of product experiments with identical components. The whole situation can be described as follows. Let $\Theta \subseteq R^k$ be a subset and let $E = (\Omega, \mathscr{A}, \{P_\vartheta\colon \vartheta \in \Theta\})$ be an experiment. Fix some $\vartheta \in \Theta$ and consider a sequence of positive numbers $\delta_n \downarrow 0$. For every $n \in N$ let

$$T_n = \{t \in R^k\colon \vartheta + \delta_n t \in \Theta\} \quad \text{and} \quad T := \lim_{n\to\infty} T_n.$$

Then one is interested in weak limits of the sequence of experiments

$$E_n = (\Omega^n, \mathscr{A}^n, \{P^n_{\vartheta+\delta_n t}\colon t \in T_n\}), \qquad n \in N.$$

The main objective of the present paper is to provide a characteristic property of all possible weak limit experiments which can be obtained in such a way. We will call this property stability of an experiment (see Definition (2.5)).

(1.6) Remark. Basically, every sequence (δ_n) can be used for rescaling. However, one may ask which reparametrizations are meaningful from a statistical point of view. The purpose of rescaling is to guarantee that the limit experiment F is non-degenerate. If $\{Q_t\colon t \in T\}$ is the family of probability measures underlying F, this means

$$0 < d(Q_s, Q_t) < 1, \qquad s \neq t.$$

Since $E_n \to F$, weakly, we have

$$\lim_{n\to\infty} \left(1 - \exp\left(n d^2(P_\vartheta, P_{\vartheta+\delta_n t})\right)\right) = d^2(Q_0, Q_t), \qquad t \in T,$$

i.e. any choice of $\delta_n = \delta_n(t)$ (possibly depending on t) which satisfies

$$n d^2(P_\vartheta, P_{\vartheta+\delta_n(t)t}) = 1$$

makes sense. If E is non-trivial and continuous, this can always be achieved. The salient point, however, is the question whether the order of the convergence of $\delta_n(t) \downarrow 0$ is independent of $t \in T$. One easily checks that

$$\delta_n(\alpha t) = \frac{1}{\alpha}\delta_n(t), \qquad \alpha > 0.$$

Hence, the order of convergence is constant on every ray starting at 0. But, in fact, there is a variety of interesting cases with order of convergence differing from ray to ray. A systematic treatment of these cases within the framework of the asymptotic theory does not exist up to now. All the situations considered so far in the literature are such that the rescaling can be done independently of the particular ray. This is not only true for the numerous standard examples where the limit is a Gaussian shift, but also for those "non-regular" cases, which are considered in Ibragimov and Has'minskii [1], Chapters V and VI. It is one of the primary aims of the present paper to start with a systematic treatment of these examples.

(1.7) Remark. Of course, even if $(\delta_n)_{n\in N}$ is an appropriate rescaling, it does not follow that $(E_n)_{n\in N}$ converges weakly. A substantial assumption of the present paper is that $(E_n)_{n\in N}$ is equicontinuous in a sense described below. As is well-known, this implies weak sequential compactness of $(E_n)_{n\in N}$. For simplicity, the theorems of this paper have been formulated for the whole sequence $(E_n)_{n\in N}$. But it is mere routine to check that they also apply to subsequences. Hence, the main theorem (2.9) yields a characteristic property of all weak accumulation points of a reparametrized equicontinuous sequence of experiments.

The paper is organized in the following way. The main results are stated and proved in the second paragraph. Section 3 contains some basic facts on translation invariant and on Gaussian experiments. In the last paragraph we have collected some consequences which are obtained by combining stability with translation invariance to give partial answers to a question posed by Müller [10].

Let us conclude the introductory section with some technical supplements.

There are several topologies on the space of probability measures which play a role in the present context. The *variational distance* between $P|\mathscr{A}$ and $Q|\mathscr{A}$ is denoted by $\|P-Q\|$. Another metric is defined by the *Hellinger distance* $d(P, Q)$ which is related with the Hellinger transform by

$$H(E_\alpha)(\tfrac{1}{2}, \tfrac{1}{2}) = 1-d^2(P_s, P_t) \quad \text{if } \alpha = \{s, t\}.$$

The topologies on $\mathscr{P}$ which are defined by these distances are identical.

For the following assume that T is a metric space with distance ϱ. An experiment $E=(\Omega, \mathscr{A}, \{P_t\colon t\in T\})$ is *continuous* if $t\mapsto P_t$ is continuous for the variational (or Hellinger) distance. Let $T_n \uparrow T$ and consider a sequence of experiments

$$E_n = (\Omega_n, \mathscr{A}_n, \{P_{n,t}\colon t\in T_n\}) \text{ in } \mathscr{E}(T_n), \qquad n\in N.$$

The sequence $(E_n)_{n\in N}$ is called *equicontinuous* if for every $\varepsilon>0$ and $t\in T$ there is a $\delta(\varepsilon, t)>0$ such that

$$\|P_{n,s}-P_{n,t}\| < \varepsilon \quad \text{if } \varrho(s, t) < \varepsilon \text{ and } \{s, t\}\subseteq T_n.$$

Equicontinuity of a sequence $(E_n)_{n\in N}$ has the following useful consequence.

Let $(\alpha_n)_{n\in N} \subseteq A(T)$ and $\alpha \in A(T)$ be such that $\alpha_n = \{t_{n,1}, \ldots, t_{n,k}\}$, $\alpha = \{t_1, \ldots, t_k\}$, and $t_{n,i} \to t_i$, $1 \leqslant i \leqslant k$. If $(E_n)_{n\in N}$ is equicontinuous and $E_n \to E$ weakly, then $H(E_{n,\alpha_n}) \to H(E_\alpha)$.

2. STABILITY OF EXPERIMENTS

Let $E_i = (\Omega_i, \mathscr{A}_i, \mathscr{P}_i)$ be experiments in $\mathscr{E}(T)$ and write $\mathscr{P}_i = \{P_{i,t}: t \in T\}$, $i = 1, 2$. Then the *product* $E_1 \otimes E_2$ is defined by $E_1 \otimes E_2 = (\Omega_1 \times \Omega_2, \mathscr{A}_1 \otimes \mathscr{A}_2, \mathscr{P})$, where $\mathscr{P} = \{P_{1,t} \otimes P_{2,t}: t \in T\}$. In an obvious manner this definition can be used to define powers of experiments.

(2.1) Definition. An experiment $E \in \mathscr{E}(T)$ is *infinitely divisible* if for every $n \in N$ there is an experiment $F_n \in \mathscr{E}(T)$ such that $F_n^n \sim E$.

If $F_n^n \sim E$, then F_n is called an *n-th root* of E. We call F_n the n-th root since it is uniquely determined up to equivalence. In an obvious manner it is possible to define rational powers of infinitely divisible experiments and it follows from weak compactness of $\mathscr{E}(T)$ that there is a weakly continuous semigroup $\lambda \mapsto E^\lambda$, $\lambda \in (0, \infty)$ (i.e. $E^{\lambda+\mu} = E^\lambda \otimes E^\mu$), such that $E_1 = E$.

The general form of the Hellinger transform of an infinitely divisible experiment is derived by LeCam [6]. An elaboration of his ideas is contained in Milbrodt and Strasser [9]. We only give important examples which will be used later.

(2.2) Examples.

(1) Let Γ be a positive definite $(k \times k)$-matrix and let $E = (\boldsymbol{R}^k, \mathscr{B}^k, \mathscr{P})$ with $\mathscr{P} = \{\nu_{t,\Gamma}: t \in \boldsymbol{R}^k\}$. If we define $E_\lambda = (\boldsymbol{R}^k, \mathscr{B}^k, \mathscr{P}_\lambda)$ with $\mathscr{P}_\lambda = \{\nu_{\lambda^{1/2}t,\Gamma}: t \in \boldsymbol{R}^k\}$, $\lambda \geqslant 0$, then easy computations show that $E_\lambda \sim E^\lambda$, $\lambda > 0$. An experiment like E is called a *Gaussian shift*. The limit experiments of Examples (1.1) and (1.3) are Gaussian shifts.

(2) Let $\mathscr{R}$ be a ring of subsets of a set X and for each $t \in T$ let μ_t: $\mathscr{R} \to [0, \infty)$ be an additive set function. Consider the experiment $E \in \mathscr{E}(T)$ whose probability measures P_t, $t \in T$, are the distributions of the Poisson processes with intensity μ_t, $t \in T$ (cf. LeCam [6] and Milbrodt [8]). Such an experiment is called a *Poisson experiment*. Every Poisson experiment E is infinitely divisible and the n-th roots are Poisson experiments with intensities μ_t/n, $t \in T$. The limit experiments of Examples (1.2) and (1.5) are Poisson experiments. This is trivial in Example (1.2) and less obvious in Example (1.5).

For the rest of the present section we assume that the parameter space T is a convex cone with vertex zero of a normed linear space and that $0 \in T$. The most important example will be the case where $T = \boldsymbol{R}^k$. It should be noted, however, that the assumption covers also the case of Example (1.2).

For the notion of stability we introduce a useful notation. If $\alpha \geqslant 0$ and $E \in \mathscr{E}(T)$, let $U_\alpha E \in \mathscr{E}(T)$ be the experiment which is obtained from E

replacing t by αt for every $t \in T$. Hence, if $E = (\Omega, \mathscr{A}, \{P_t\colon t \in T\})$, then $U_\alpha E = (\Omega, \mathscr{A}, \{P_{\alpha t}\colon t \in T\})$.

(2.3) LEMMA. *The system* $(U_\alpha)_{\alpha \geqslant 0}$ *is a semigroup of weakly continuous transformations with the following properties*:

(1) $U_\alpha U_\beta = U_{\alpha\beta}$, *where* $\alpha \geqslant 0$, $\beta \geqslant 0$.

(2) $U_\alpha E \otimes U_\alpha F = U_\alpha(E \otimes F)$ *for* $\alpha \geqslant 0$.

(3) $E \sim F$ *implies* $U_\alpha E \sim U_\alpha F$ *for* $\alpha \geqslant 0$.

(4) *If* E *is continuous at zero, then* $\lim_{\alpha \to 0} U_\alpha E$ *is trivial.*

(5) *If* E *is continuous at zero and non-trivial, then* $U_\alpha E \sim U_\beta E$ *implies* $\alpha = \beta$.

Proof. Assertions (1)-(4) are obvious. To prove (5) let $U_\alpha E \sim U_\beta E$. This implies $E \sim U_{(\beta/\alpha)^k} E$ for every $k \in N$. If $\alpha > \beta$, it follows that $(\beta/\alpha)^k \to 0$ as $k \to \infty$ and continuity of E at zero implies that E is trivial. Hence $\alpha \leqslant \beta$. For reasons of symmetry it follows that $\alpha = \beta$.

Condition (4) of the following theorem coincides with the notion of scale invariance introduced by Müller [10].

(2.4) THEOREM. *Let* $E \in \mathscr{E}(T)$ *be a non-trivial continuous experiment. The following properties are equivalent*:

(1) *For all* $\alpha > 0$, $\beta > 0$, *there is a* $\gamma > 0$ *such that* $U_\alpha E \otimes U_\beta E \sim U_\gamma E$.

(2) *For every* $n \in N$ *there is a* $c_n > 0$ *such that* $E^n \sim U_{c_n} E$.

(3) *There is some* $p > 0$ *such that* $E^n \sim U_{n^{1/p}} E$ *for every* $n \in N$.

(4) E *is infinitely divisible, and there is some* $p > 0$ *such that* $E^\alpha \sim U_{\alpha^{1/p}} E$ *for every* $\alpha \geqslant 0$.

Proof. (1)$\Rightarrow$(2). Let $c_1 = 1$ and define $(c_n)_{n \in N}$ inductively by $U_{c_n} E \otimes E \sim U_{c_{n+1}} E$, $n \in N$. It is obvious that $U_{c_n} E \sim E^n$ for every $n \in N$.

(2)$\Rightarrow$(3). For convenience $c_n =: c(n)$, $n \in N$. For m, $n \in N$ we have

$$\begin{aligned} U_{c(mn)} E \sim E^{mn} = (E^m)^n \sim (U_{c(m)} E)^n \\ = U_{c(m)}(E^n) \sim U_{c(m)} U_{c(n)} E = U_{c(m)c(n)} E. \end{aligned}$$

It follows from Lemma (2.3), (5), that $c(mn) = c(m)c(n)$, where m, $n \in N$.

Let us show that $n \mapsto c(n)$ is increasing. Assuming the contrary, let $c(r+1) < c(r)$ for some $r \in N$. If $a := c(r+1)/c(r) < 1$, it follows from continuity of E that

$$\lim_{k \to \infty} d(P_0, P_{a^k t}) = 0, \quad t \in T.$$

On the other hand, noting that $a^k = c\big((r+1)^k\big)/c(r^k)$, we obtain

$$(U_{a^k} E)^{r^k} = U_{a^k}(E^{r^k}) \sim U_{a^k} U_{c(r^k)} E \sim U_{c((r+1)^k)} E \sim E^{(r+1)^k}$$

which implies that

$$1 - d^2(P_0, P_{a^k t}) = \big(1 - d^2(P_0, P_t)\big)^{[(r+1)/r]^k}.$$

Hence, for every $t \in T$ with $d(P_0, P_t) > 0$, we have

$$\lim_{k \to \infty} d(P_0, P_{a^k t}) = 1$$

which is the desired contradiction.

If $m, n \in N$ are such that $m > n > 1$, then for each $k \in N$ there is an $l(k) \in N$ such that

$$n^{l(k)} \leqslant m^k < n^{l(k)+1},$$

which implies

$$c^{l(k)}(n) \leqslant c^k(m) \leqslant c^{l(k)+1}(n).$$

Combining these inequalities and taking logarithms we obtain

$$\frac{l(k)}{l(k)+1} \frac{\log c(n)}{\log n} \leqslant \frac{\log c(m)}{\log m} \leqslant \frac{l(k)+1}{l(k)} \frac{\log c(n)}{\log n};$$

for $k \to \infty$ this yields

$$\frac{\log c(n)}{\log n} = \frac{\log c(m)}{\log m} =: \alpha \geqslant 0.$$

The number $\alpha \geqslant 0$ is independent of $m, n \in N$ which implies that $c_n = n^\alpha$, $n \in N$. It remains to show that $\alpha > 0$. If $\alpha = 0$, then $E^n \sim E$ for every $n \in N$, which implies

$$1 - d^2(P_0, P_t) = \big(1 - d^2(P_0, P_t)\big)^n, \qquad t \in T.$$

Hence $d(P_0, P_t)$ can assume only the values 0 and 1 which is a contradiction to the continuity of E.

(3) $\Rightarrow$ (4). To prove that E is infinitely divisible we observe that for every $n \in N$

$$(U_{n^{-1/p}} E)^n = U_{n^{-1/p}}(E^n) \sim U_{n^{-1/p}} U_{n^{1/p}} E = E$$

which implies that $E^{1/n} \sim U_{n^{-1/p}} E$.

Let $\alpha \geqslant 0$ and choose $\big(k(n)\big)_{n \in N}$ such that $\lim \big(k(n)/n\big) = \alpha$. Then continuity implies

$$\lim_{n \to \infty} U_{(k(n)/n)^{1/p}} E = U_{\alpha^{1/p}} E, \text{ weakly.}$$

However, (3) implies

$$U_{(k(n)/n)^{1/p}} E = U_{n^{-1/p}} U_{k(n)^{1/p}} E \sim U_{n^{-1/p}}(E^{k(n)})$$
$$\sim (U_{n^{-1/p}} E)^{k(n)} \sim E^{k(n)/n},$$

which proves that $E^\alpha \sim U_{\alpha^{1/p}} E$.

(4) ⇒ (1). This is obvious by the relation

$$U_\alpha E \otimes U_\beta E \sim E^{\alpha^p} \otimes E^{\beta^p} \sim E^{\alpha^p+\beta^p} \sim U_{(\alpha^p+\beta^p)^{1/p}} E, \qquad \alpha > 0,\ \beta > 0.$$

(2.5) Definition. A continuous experiment $E \in \mathscr{E}(T)$ is *stable* if it is either trivial or if any of the equivalent conditions (1)-(4) in (2.4) is satisfied.

If $E \in \mathscr{E}(T)$ is stable and non-trivial, then the exponent p of conditions (3) or (4) in (2.4) is called the *characteristic exponent.*

(2.6) LEMMA. *A continuous, non-trivial experiment $E \in \mathscr{E}(T)$ is stable with exponent $p > 0$ iff the logarithms of its Hellinger transforms*

$$(t_1, \ldots, t_k) \mapsto H(E_{\{t_1 \ldots t_k\}}), \qquad (t_1 \ldots t_k) \in T^k,\ k \in N,$$

are homogeneous functions of degree p.

Proof. This is an immediate consequence of (2.4) (4).

(2.7) Examples.

(1) The limit experiments of Examples (1.1)-(1.5) are stable.

(2) A non-trivial Poisson experiment E whose intensities are uniformly bounded measures cannot be stable since

$$\log H(E_{\{s,t\}})(\tfrac{1}{2}, \tfrac{1}{2}) = -d^2(\mu_s, \mu_t)$$

is bounded on $T \times T$.

Now, we show that the class of stable experiments coincides with the class of weak limits of equicontinuous sequences $(U_{\delta_n} E^n)_{n \in N}$, where $\delta_n \downarrow 0$. This extends the suggestion of Müller [10], in that we do not specify the dependence of δ_n on $n \in N$.

First we note some simple facts.

(2.8) Remark. Let $E \in \mathscr{E}(T)$ and $\delta_n \downarrow 0$. Assume that $(U_{\delta_n} E^n)_{n \in N}$ is equicontinuous and converges weakly to a limit $F \in \mathscr{E}(T)$. If F is not trivial, then

(1) $\lim_{n \to \infty} (\delta_{n+1}/\delta_n) = 1$;

(2) for every $m \in N$ the sequence $(\delta_{mn}/\delta_n)_{n \in N}$ converges in $(0, 1]$.

For the easy proofs note that the sequences have accumulation points in $[0, 1]$ and apply elementary limiting properties of the Hellinger distances.

(2.9) THEOREM. *An experiment $F \in \mathscr{E}(T)$ is stable iff it is the weak limit of an equicontinuous sequence $(U_{\delta_n} E^n)_{n \in N}$, where $E \in \mathscr{E}(T)$ and $\delta_n \downarrow 0$.*

Proof. First assume that F is stable. If F is trivial, then the assertion is obvious. If F is not trivial but stable with exponent $p > 0$, then we define $\delta_n = n^{-1/p}$ and observe that $E_n = U_{\delta_n} F^n$ satisfies $E_n \sim F$ for every $n \in N$. Write $F = (\Omega_0, \mathscr{A}_0, \{Q_t \colon t \in T\})$. Then we have $d(Q^n_{\delta_n s}, Q^n_{\delta_n t}) = d(Q_s, Q_t)$, $s, t \in T$, which proves the necessity of the condition.

Conversely, assume that there are an experiment $E \in \mathscr{E}(T)$ and $\delta_n \downarrow 0$ such that $(U_{\delta_n} E^n)_{n\in N}$ is equicontinuous and $U_{\delta_n} E^n \to F$ weakly. In view of Remark (2.8), (2), for every $m \in N$ there exists

$$c_m = \lim_{n\to\infty} \frac{\delta_{mn}}{\delta_n}.$$

Then

$$F = \lim_{n\to\infty} U_{\delta_{mn}} E^{mn} = \lim_{n\to\infty} U_{c_m} U_{\delta_n} E^{nm}$$
$$= U_{c_m} (\lim_{n\to\infty} U_{\delta_n} E^n)^m = U_{c_m} F^m.$$

Now, Theorem (2.4) completes the proof.

If $\delta_n = n^{-1/p}$, $n \in N$, the preceding assertion is trivial in that it does not rely on Theorem (2.4). But this is not necessarily the case, as Example (1.4) shows. There, for $\alpha = 1/2$, the stability exponent is $p = 2$ but $\delta_n = (n \log n)^{-1/2}$, $n \in N$. The sequence (δ_n) cannot be replaced by $\delta_n = n^{-1/2}$, $n \in N$, since this choice would lead to a degenerate limit experiment. The general form of the sequence (δ_n) is given below.

(2.10) Corollary. *Assume that $(U_{\delta_n} E^n)_{n\in N}$ is an equicontinuous sequence converging weakly to a non-trivial $F \in \mathscr{E}(T)$. Then*

$$\delta_n = n^{-1/p} a_n, \qquad n \in N,$$

where $p > 0$ is the exponent of stability of F and

$$\lim_{n\to\infty} \frac{a_{nm}}{a_n} = 1 \qquad \textit{for every } m \in N.$$

Proof. Let $m \in N$. Since

$$\left(\frac{a_{mn}}{a_n}\right)^p = \left(\frac{\delta_{mn}}{\delta_n}\right)^p \cdot m, \qquad n \in N,$$

we have to show that

$$\lim_{n\to\infty} \frac{\delta_{mn}}{\delta_n} = \left(\frac{1}{m}\right)^{1/p}.$$

Let c_m be the limit of $(\delta_{mn}/\delta_n)_{n\in N}$ (cf. Remark (2.8), (2)). If $t \in T$ is such that $0 < d(Q_0, Q_t) < 1$, then

$$\lim_{n\to\infty} mnd^2(P_0, P_{\delta_{mn}t}) = -\log\left(1 - d^2(Q_0, Q_t)\right)$$

and

$$\lim_{n\to\infty} mnd^2(P_0, P_{\delta_{mn}t}) = \lim_{n\to\infty} mnd^2(P_0, P_{\delta_n c_m t})$$
$$= -m \log\left(1 - d^2(Q_0, Q_{c_m t})\right).$$

From Lemma (2.6) we get

$$\log\left(1-d^2(Q_0, Q_{c_m t})\right) = c_m^p \log\left(1-d^2(Q_0, Q_t)\right),$$

which implies $mc_m^p = 1$.

The property the sequence of factors $(a_n)_{n\in N}$ inherits according to the preceding assertion is slow variation for $n\to\infty$; in particular,

$$\lim_{n\to\infty} \frac{a_n}{n^\varepsilon} = 0 \quad \text{and} \quad \lim_{n\to\infty} n^\varepsilon a_n = \infty, \qquad \varepsilon > 0.$$

Examples of such sequences are logarithms and powers of logarithms.

The preceding results explain why the limit experiments of examples (1.4) and (1.5) must be stable and clarify the particular form of the rescaling constants $\delta_n \downarrow 0$. The following obvious modification of the argument extends the result also to a more general situation, which covers Examples (1.1)-(1.3):

(2.11) Remark. Let $D \subseteq T$ be a closed subset which is starshaped in the following sense: the set D contains 0, and every ray $\lambda \mapsto \lambda t$, $t\in T$, intersects ∂D at exactly one point $\lambda > 0$. Let $E_0 = (\Omega, \mathscr{A}, \{P_t\colon t\in D\})$ be a continuous experiment. Then we may construct a continuous experiment $E = (\Omega, \mathscr{A}, \{P_t\colon t\in T\})$ with $E_D = E_0$ by the definition $P_t = P_{\lambda_t t}$ if $\lambda_t = \sup\{\lambda\in[0, 1]\colon \lambda t\in D\}$, $t\in T$. Let $\delta_n\downarrow 0$ and $T_n = \{t\in T\colon \delta_n t\in D\}$. Then the weak limits of the sequence of experiments

$$E_n = (\Omega^n, \mathscr{A}^n, \{P^n_{\delta_n t}\colon t\in T_n\}), \qquad n\in N,$$

are the same as the weak limits of $(U_{\delta_n} E^n)_{n\in N}$.

3. TRANSLATION INVARIANT EXPERIMENTS AND GAUSSIAN EXPERIMENTS

In this paragraph we collect and prove, if necessary, some facts on translation invariance and Gaussian experiments.

(3.1) Definition. Assume that T is a linear space. An experiment $E = (\Omega, \mathscr{A}, \mathscr{P} = \{P_t\colon t\in T\}) \in \mathscr{E}(T)$ is *translation invariant* if E is equivalent to $(\Omega, \mathscr{A}, \{P_{t+s}\colon t\in T\})$ for every $s\in T$.

The importance of translation invariance is due to the fact that in many situations almost every limit experiment is translation invariant. This has been shown by LeCam [5].

A simple description of translation invariance is as follows. E is translation invariant iff, for every $s\in T$ and $r\in T$,

$$\mathscr{L}\left(\left(\frac{dP_t}{dP_r}\right)_{t\in T} | P_r\right) = \mathscr{L}\left(\left(\frac{dP_{t+s}}{dP_{r+s}}\right)_{t\in T} | P_{r+s}\right).$$

An obvious consequence of translation invariance of an experiment E is the translation invariance of the Hellinger distances between the probability measures of E. The limit experiments of examples (1.1) and (1.3)-(1.5) are translation invariant.

(3.2) Definition. An experiment $E \in \mathscr{E}(T)$ is a *Gaussian experiment* if it is homogeneous and if the loglikelihood process $(\log (dP_t/dP_0))_{t\in T}$ is a Gaussian process.

(3.3) Remark. The following facts are easy to prove and well-known. The property of being a Gaussian experiment is a property of equivalence classes of experiments. The equivalence class is uniquely determined by the covariance $K\colon T\times T\to \mathbf{R}$ of the loglikelihood process $(\log (dP_t/dP_0))_{t\in T}$ which is a positive semidefinite kernel satisfying $K(s, 0) = K(0, t) = 0$ for all $s\in T$, $t\in T$. The covariance K and the Hellinger distances determine each other by the formulas

$$1-d^2(P_s, P_t) = \exp\left[\tfrac{1}{4}\left(K(s, t)-\frac{K(s, s)+K(t, t)}{2}\right)\right]$$

and

$$K(s, t) = 4\big(a(s, 0)+a(0, t)-a(s, t)\big)$$

if $a(s, t) = -\log\big(1-d^2(P_s, P_t)\big)$.

This is a consequence of the general form of the Hellinger transform of Gaussian experiments. Hence, it follows that the function $(s, t)\mapsto d(P_s, P_t)$, $(s, t)\in T\times T$, determines the equivalence class of a Gaussian experiment.

(3.4) Theorem. *Assume that T is a linear space. Let $E = (\Omega, \mathscr{A}, \{P_t\colon t\in T\})$ be a Gaussian experiment. Then each of the following conditions is equivalent to translation invariance of E:*

(1) $d(P_{t_1}, P_{t_2}) = d(P_{t_1+s}, P_{t_2+s})$ *for all* $t_1, t_2, s\in T$.

(2) *The centered loglikelihood process*

$$\left(\log\frac{dP_t}{dP_0}-P_0\left(\log\frac{dP_t}{dP_0}\right)\right)_{t\in T}$$

has stationary increments.

Proof. (1) The preceding remark shows that two Gaussian experiments are equivalent iff the Hellinger distances between corresponding pairs of probability measures are equal. This implies that translation invariance is equivalent to condition (1).

(2) We keep the notations of Remark (3.3) and note that translation invariance of E is equivalent to translation invariance of the function $a\colon T\times T\to \mathbf{R}$. Let us write

$$X_t = \log\frac{dP_t}{dP_0}, \qquad t\in T.$$

Then we observe that if $t \in T$, $h \in T$,

$$\mathrm{Cov}\,(X_{s+h}-X_s,\ X_{t+h}-X_t)$$
$$= 4\big(a(s+h,\ t)+a(s,\ t+h)-a(s+h,\ t+h)-a(s,\ t)\big).$$

Hence, translation invariance of E implies stationary increments of $(X_t)_{t\in T}$. The converse follows from

$$V(X_{t+h}-X_t) = 8a(t,\ t+h), \qquad t\in T,\ h\in T.$$

It is clear that there are as many translation invariant Gaussian experiments as there are centered Gaussian processes with stationary increments. The following examples are typical limit experiments.

(3.5) Examples.

(1) The experiment E of Example (2.2), (1), is the most simple Gaussian experiment. It is translation invariant. The loglikelihood process is of the form $X_t = t'X - \frac{1}{2}t'\Gamma^{-1}t$, where X is a random variable satisfying $\mathscr{L}(X|P_0) = \nu_{0,\Gamma}$. The covariance structure is $K(s,\ t) = s'\Gamma^{-1}t$, $s\in \boldsymbol{R}^k$, $t\in \boldsymbol{R}^k$.

(2) The limit experiments of Example (1.4) are translation invariant Gaussian experiments for $T = \boldsymbol{R}$ with covariance $K_\varrho(s,\ t) = \frac{1}{2}(|s|^\varrho + |t|^\varrho - |s-t|^\varrho)$, where $1 < \varrho \leqslant 2$. The case $\varrho = 2$ is the trivial shift of the preceding example. In the cases $1 < \varrho < 2$, however, the loglikelihood processes cannot be put into a similar simple form.

Example (3.5), (1), is a finite-dimensional Gaussian shift. It is well-known that every Gaussian experiment is equivalent to a subexperiment of a Gaussian shift, where the underlying Hilbert space may be of infinite dimension. Let us collect the basic facts in the following discussion.

(3.6) Discussion. The following is taken from LeCam [7]. Let H be a Hilbert space and $(\Omega_0,\ \mathscr{A}_0,\ Q_0)$ a probability space. Any linear mapping $L\colon H \to L_2(\Omega_0,\ \mathscr{A}_0,\ Q_0)$ is a linear process. If

$$\mathscr{L}\big(L(z)|Q_0\big) = \nu_{\langle x,z\rangle,\|z\|^2}, \qquad z\in H,$$

then L is a standard linear Gaussian process with mean $x\in H$. An experiment $F = (\Omega_0,\ \mathscr{A}_0,\ \{Q_x\colon x\in H\})$ is a standard Gaussian shift if

$$\frac{dQ_x}{dQ_0} = \exp\big(L(x) - \tfrac{1}{2}\|x\|^2\big), \qquad x\in H,$$

where L is a standard linear Gaussian process with mean 0. Let $E = (\Omega,\ \mathscr{A},\ \{P_t\colon t\in T\})$ be an arbitrary Gaussian experiment. If K is the covariance of E, then there is a Hilbert space H and a mapping $\psi\colon T\to H$ such that $K(s,\ t) = \langle\psi(s),\ \psi(t)\rangle$, $s\in T$, $t\in T$. If F is the standard Gaussian shift on H, then E is equivalent with that subexperiment of F which is defined by the family of probability measures $\{Q_{\psi(t)}\colon t\in T\}$. This is due to the fact that the latter is a Gaussian experiment with covariance K.

(3.7) Example. The experiments of Example (3.5), (2), can be embedded into a Gaussian shift in the following way. Let $H = L_2(\mathbf{R}, \lambda)$ and $\psi_\alpha(t)$: $x \mapsto |x-t|^\alpha - |x|^\alpha$, $x \in \mathbf{R}$, where $\varrho = 2\alpha+1$, $-\frac{1}{2} < \alpha < \frac{1}{2}$. Then easy computations show that $K_\varrho(s, t) = \langle \psi_\alpha(s), \psi_\alpha(t) \rangle$, $s \in \mathbf{R}$, $t \in \mathbf{R}$, for $0 < \varrho < 2$. Moreover, we observe that these examples cannot be embedded into a finite-dimensional shift since $\{\psi_\alpha(t)\colon t \in \mathbf{R}\}$ is a linearly independent set for every $\alpha \in (-\frac{1}{2}, \frac{1}{2})$. For further discussion of this example cf. Pflug [11].

Let us state a simple characterization of those curves $\psi\colon \mathbf{R}^k \to H$ which define a translation invariant Gaussian experiment. Obviously, these are exactly the curves which define Gaussian processes with stationary increments (cf. Kolmogorov [3]). The following theorem will be needed to obtain a characterization of one-dimensional Gaussian shifts by their statistical properties. We call a mapping $U\colon H \to H$ a *motion* if it is a composition of a linear isometry and a translation.

(3.8) Theorem. *Assume that T is a linear space. A subexperiment $E \in \mathscr{E}(T)$ of a standard Gaussian shift on a Hilbert space H is translation invariant iff the defining curve $\psi\colon T \to H$ satisfies any of the following conditions*:

(1) *for every pair* $s \in T$, $t \in T$,

$$\|\psi(s) - \psi(t)\| = \|\psi(s-t)\|;$$

(2) *there is a semi-group* $\{U_t\colon t \in T\}$ *of motions on* H *such that* $\psi(t) = U_t(0)$, $t \in T$.

Proof. (1) Let us show that condition (1) is equivalent to translation invariance. Easy computations show that the standard Gaussian shift satisfies

$$d^2(Q_x, Q_y) = 1 - \exp\left(-\frac{1}{8}\|x-y\|^2\right), \qquad x \in H,\ y \in H.$$

Hence the assertion follows from Theorem (3.4), (1).

(2) The second condition implies the first in view of

$$\|\psi(s) - \psi(t)\| = \|U_s(0) - U_t(0)\| = \|U_s(0) - U_s U_{t-s}(0)\|$$
$$= \|U_{t-s}(0)\| = \|\psi(t-s)\|.$$

Assume conversely that condition (1) is satisfied. We define a group $\{A_t\colon t \in T\}$ of isometries such that $U_t x = A_t x + \psi(t)$, $t \in T$, is the desired semi-group of motions. Let

$$A_t \psi(h) = \psi(t+h) - \psi(t), \qquad h, t \in T.$$

To show that this defines linear maps A_t on span $\{\psi(h)\colon h \in T\}$ we note first that

$$\langle \psi(t+h_1) - \psi(t), \psi(t+h_2) - \psi(t) \rangle = \langle \psi(h_1), \psi(h_2) \rangle, \qquad t \in T,\ h_i \in T,\ i = 1, 2.$$

This follows easily from condition (1).

Hence for arbitrary choices $\lambda_i \in \boldsymbol{R}$, $h_i \in T$, $1 \leqslant i \leqslant n$, the relation

$$\sum_{i=1}^{n} \lambda_i \psi(h_i) = 0$$

implies

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j \langle \psi(h_i), \psi(h_j) \rangle = 0$$

and, therefore,

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j \langle \psi(t+h_i) - \psi(t), \psi(t+h_j) - \psi(t) \rangle = 0,$$

which implies

$$\sum_{i=1}^{n} \lambda_i \big(\psi(t+h_i) - \psi(t)\big) = 0.$$

Thus A_t can be extended to linear maps on span $\{\psi(h)\colon h \in T\}$. From condition (1) it is obvious that the A_t, $t \in T$, are isometries. The set $\{A_t\colon t \in T\}$ is a semi-group of isometries since

$$\begin{aligned} A_s A_t \psi(h) &= A_s\big(\psi(t+h) - \psi(t)\big) = A_s \psi(t+h) - A_s \psi(t) \\ &= \psi(t+s+h) - \psi(s) - \big(\psi(t+s) - \psi(s)\big) \\ &= \psi(t+s+h) - \psi(t+s) = A_{s+t} \psi(h). \end{aligned}$$

It is clear that each A_t can be extended to H such that $\{A_t\colon t \in T\}$ remains a semi-group of isometries. The equations $\psi(t) = U_t(0)$, $t \in T$, are trivial. It remains to show that $\{U_t\colon t \in T\}$ is a semi-group of motions in H. This follows from

$$\begin{aligned} U_s U_t x &= U_s\big(A_t x + \psi(t)\big) = A_s A_t x + A_s \psi(t) + \psi(s) \\ &= A_{s+t} x + \psi(t+s) - \psi(s) + \psi(s) = U_{s+t} x, \qquad x \in H. \end{aligned}$$

(3.9) Discussion. Assume that $T = \boldsymbol{R}$ and $E \in \mathscr{E}(\boldsymbol{R})$ is a translation invariant and continuous Gaussian experiment with Hilbert-space representation $\psi\colon \boldsymbol{R} \to H$. It is possible to characterize the one-dimensional shift by statistical properties. Let $\varphi_{t,\alpha}$ be the NP-test of level $\alpha \in (0, 1)$ for the testing problem $H = \{P_0\}$ against $K = \{P_t\}$. Then easy computations yield the power function

$$P_s(\varphi_{t,\alpha}) = \Phi\big(N_\alpha + \|\psi(s)\| \cos\big(\psi(s), \psi(t)\big)\big), \qquad s \in \boldsymbol{R},\ t \in \boldsymbol{R},$$

where Φ is the distribution function of $\nu_{0,1}$ and N_α is the α-quantile of $\nu_{0,1}$.

Assume that there is a test φ_0 such that $P_0 \varphi_0 = \alpha$ and

$$P_s \varphi_0 = \begin{cases} \sup \{P_s \varphi\colon\ P_0 \varphi = \alpha\} & \text{if } s \geqslant 0, \\ \inf \{P_s \varphi\colon\ P_0 \varphi = \alpha\} & \text{if } s \leqslant 0, \end{cases}$$

i.e. a uniformly most powerful test of level α for the testing problem $H = \{P_s\colon\ s \leqslant 0\}$ against $K = \{P_s\colon\ s > 0\}$. We will show that in this case E is necessarily a one-dimensional Gaussian shift. Indeed, on the one hand, we have

$$P_s \varphi_0 = \Phi(N_\alpha + \|\psi(s)\| \operatorname{sgn} s) \quad \text{if } s \in \boldsymbol{R}$$

and, on the other hand, $\varphi_0 = \varphi_{t,\alpha}$ P_s-a.e. for every $s \in \boldsymbol{R}$, $t \in \boldsymbol{R}$, and hence

$$P_s \varphi_0 = \Phi\big(N_\alpha + \|\psi(s)\| \cos\big(\psi(t), \psi(s)\big) \operatorname{sgn} t\big), \quad \text{if } s \in \boldsymbol{R} \text{ and } t \in \boldsymbol{R},$$

which implies that $\cos\big(\psi(s), \psi(t)\big) = \operatorname{sgn} s \cdot \operatorname{sgn} t$ for all $s \in \boldsymbol{R}$, $t \in \boldsymbol{R}$, and, therefore, the image of ψ in H is a straight line. It follows that $\psi(t) = f(t)\psi(1)$, $t \in \boldsymbol{R}$, and $f\colon \boldsymbol{R} \to \boldsymbol{R}$ is additive. From continuity it follows that $f(t) = ct$, $t \in \boldsymbol{R}$, which proves the assertion.

4. SOME CONSEQUENCES OF STABILITY

Müller [10] asked the question for the totality of restrictions which are induced by translation and stability together. Since every stable experiment is infinitely divisible, and since every infinitely divisible experiment admit a unique decomposition into a Gaussian and a Poisson part, the problem is two-fold. It is answered by characterizing separately the translation and stable Gaussian experiment and the translation invariant and stable Poisson experiments. Here, we consider the first part of the problem.

(4.1) THEOREM. *Assume that T is a linear space. If $E \in \mathscr{E}(T)$ is a Gaussian experiment which is translation and stable, then the characteristic exponent p satisfies $p \leqslant 2$. If $T = \boldsymbol{R}$, then the covariance K is of the form*

$$K(s, t) = c\,|s|^p + |t|^p - |s-t|^p), \qquad s, t \in \boldsymbol{R}.$$

Proof. At first consider the case $k = 1$. If $f(s) := -\log\big(1 - d^2(P_0, P_s)\big)$, $s \in \boldsymbol{R}$, then we obtain from Remark (3.3) in view of translation invariance that

$$K(s, t) = 4\big(f(s) + f(t) - f(s-t)\big), \qquad s, t \in \boldsymbol{R}.$$

From translation invariance we obtain also that $f(s) = f(-s)$, $s \in \boldsymbol{R}$. Stability of order $p > 0$ implies that $f(s) = s^p f(1)$, $s \geqslant 0$, which proves the second assertion. The exponent p must satisfy $p \leqslant 2$ since otherwise K is not positive semi-definite. If $k > 1$, then the exponent p also satisfies $p \leqslant 2$ since the first part of the proof may be applied to any one-dimensional subexperiment of E.

It should be noted that for every $p \in (0, 2]$ the function K, as defined in the preceding theorem, is a covariance. This follows from the representation given in Example (3.7).

(4.2) Remark. *The characterization of the covariance structures of translation and stable Gaussian experiments also yields a characterization of the appertaining Hellinger transforms. In case $T = \boldsymbol{R}^k$, $k > 1$, a characterization of the covariance structure analogous to the case $k = 1$ can be given if the limiting experiment is invariant under orthogonal transformations.*

Starting from the present paper the interesting problem, whether a similarly complete description of the class of translation and stable Poisson experiments can be given, has been attacked. Janssen [2] proved that the characteristic exponent p of any translation and stable Poisson experiment satisfies $p \leqslant 2$. For the important subclass of Poisson experiments with independent increments a complete description of its translation and stable elements is contained in Strasser [13].

Let $E = (\Omega, \mathscr{A}, \{P_t\colon t \in \boldsymbol{R}^k\})$ be stable. Müller [10] claims that the exponent of stability $p = 2$ implies that the binary subexperiments (P_s, P_t), $(s, t) \in \boldsymbol{R}^k \times \boldsymbol{R}^k$, are Gaussian, provided E satisfies regularity conditions. Unfortunately, those regularity conditions are not specified. Without any regularity conditions the assertion is obviously wrong. We show below that imposing translation invariance leads to a far stronger conclusion, namely that E is even a Gaussian shift experiment. This is obtained by Müller [10] under the additional hypothesis that E is an exponential family.

(4.3) Theorem. *Assume that $T = \boldsymbol{R}^k$. Let $E \in \mathscr{E}(T)$ be a continuous experiment which is translation and stable with exponent p. Then $p = 2$, iff E is a standard Gaussian shift of dimension k.*

Proof. Assume that $p = 2$. For the Hellinger distances we have

$$d^2(P_t, P_{t+h}) = 1 - \exp\left(-f(h)\right), \qquad t \in \boldsymbol{R}^k,\ h \in \boldsymbol{R}^k,$$

where $f\colon \boldsymbol{R}^k \to \boldsymbol{R}$ is a continuous function satisfying $f(h) = \|h\|^2 f(h/\|h\|)$, $h \in \boldsymbol{R}^k$. Since f is bounded on the unit sphere of $\boldsymbol{R}^k$, it follows that

$$\overline{\lim_{h \to 0}} \frac{d^2(P_t, P_{t+h})}{\|h\|^2} < \infty \qquad \text{if } t \in \boldsymbol{R}^k.$$

Now the results of LeCam [4] imply that for Lebesgue-almost every $s \in \boldsymbol{R}^k$ the experiments $(\Omega^n, \mathscr{A}^n, \mathscr{P}_{n,s})$ with

$$\mathscr{P}_{n,s} = \{P^n_{s+tn^{-1/2}}\colon t \in \boldsymbol{R}^k\}, \qquad n \in N,$$

converge weakly to a standard Gaussian shift of dimension k. In view of translation invariance this is even true for every $s \in \boldsymbol{R}^k$, in particular for $s = 0$. From stability we obtain that $(\Omega^n, \mathscr{A}^n, \mathscr{P}_{n,0}) \sim E$ for every $n \in N$, which proves the assertion.

Thus, only Gaussian shifts in $\mathscr{E}(\boldsymbol{R}^k)$ can be translation invariant and stable with exponent $p = 2$. For the exponents $0 < p < 2$ there are both Gaussian as well as non-Gaussian experiments being translation invariant and stable with this exponent. According to Discussion (3.6) these Gaussian experiments can be embedded as subexperiments of standard Gaussian shifts. It turns out that for $p < 2$ the underlying Hilbert space cannot be of finite dimension.

(4.4) THEOREM. *Assume that T is a linear space. Let $E \in \mathscr{E}(T)$ be a Gaussian experiment which is translation invariant and stable with exponent $p < 2$. Then E cannot be embedded into a finite-dimensional Gaussian shift.*

Proof. The assertion is proved if we show that it is true for the case $T = \boldsymbol{R}$. In this case we have a complete survey of all possible covariances by Theorem (4.1). Hence the arguments of Example (3.7) may be applied.

Finally, we consider a statistical implication of stability. From Discussion (3.9) we have seen that in general NP-tests need not be uniformly most powerful if $E \in \mathscr{E}(\boldsymbol{R})$ is a translation invariant and stable Gaussian experiment. This means that in a common asymptotic framework sequences of NP-tests need not be asymptotically uniformly most powerful. We would like to compute relative efficiencies of various sequences of NP-tests.

(4.5) Discussion. Assume that $T = \boldsymbol{R}$. Let us consider the situation which is described adjacent to Example (1.5). Assume that there is a weak limit experiment F which is a non-trivial Gaussian experiment being translation invariant and stable with exponent p. The Hilbert-space representation of F is given by $\psi\colon \boldsymbol{R} \to H$. Let $(\varphi_n)_{n \in N}$ be a sequence of NP-tests of level α for $H_n = \{P_\vartheta^n\}$ against $K_n = \{P_{\vartheta+\delta_n s}^n\}$ and let $(\bar{\varphi}_n)_{n \in N}$ be a sequence of NP-tests of level α for $H_n = \{P_\vartheta^n\}$ against $K_n = \{P_{\vartheta+\delta_n t}^n\}$, $s > 0$, $t > 0$. We know that

$$\lim_{n \to \infty} P_{\vartheta+\delta_n s}^n \varphi_n = \Phi\big(N_\alpha + \|\psi(s)\|\big)$$

and that

$$\lim_{n \to \infty} P_{\vartheta+\delta_n s}^n \bar{\varphi}_n = \Phi\big(N_\alpha + \|\psi(s)\| \cos\big(\psi(s), \psi(t)\big)\big).$$

Hence the asymptotic power of $(\bar{\varphi}_n)$ at $t \in \boldsymbol{R}$ may be smaller than that of (φ_n) and we need larger sample sizes to reach the same power with $(\bar{\varphi}_n)$.

Let $(k(n))_{n \in N} \subseteq N$ be any sequence such that

$$\lim_{n \to \infty} P_{\vartheta+\delta_n s}^{k(n)} \bar{\varphi}_{k(n)} = \Phi\big(N_\alpha + \|\psi(s)\|\big).$$

It is not clear that there are such sequences at all. But if such a sequence exists and if there is

$$\mu := \lim_{n \to \infty} \frac{k(n)}{n},$$

then it is natural to identify $1/\mu$ with the relative Pitman efficiency of $(\bar{\varphi}_n)$ with respect to (φ_n) at $t \in \boldsymbol{R}$.

Let us show that $\mu = 1$ can only happen if $p = 2$. First we observe that

$$\lim_{n \to \infty} P^{k(n)}_{\vartheta + \delta_{k(n)}(\delta_n/\delta_{k(n)})s} \bar{\varphi}_{k(n)} = \Phi\big(N_\alpha + \|\psi(s)\|\big).$$

Since $0 < \Phi\big(N_\alpha + \|\psi(s)\|\big) < 1$, it follows that $(\delta_n/\delta_{k(n)})_{n \in \boldsymbol{N}}$ keeps bounded away from 0 and ∞. If λ is any limit point of $(\delta_n/\delta_{k(n)})_{n \in \boldsymbol{N}}$, then we have necessarily $\lambda = 1$ and, therefore,

$$\lim_{n \to \infty} P^{k(n)}_{\vartheta + \delta_{k(n)}(\delta_n/\delta_{k(n)})s} \bar{\varphi}_{k(n)} = \Phi\big(N_\alpha + \|\psi(s)\| \cos\big(\psi(s), \psi(t)\big)\big)$$

which implies

$$\|\psi(s)\| = \|\psi(s)\| \cos\big(\psi(s), \psi(t)\big)$$

or explicitly

$$|s|^p + |t|^p - |s-t|^p = 2|s|^{p/2}|t|^{p/2}.$$

It follows that

$$(|s|^{p/2} - |t|^{p/2})^2 = |s-t|^p,$$

which can only be satisfied for $s \neq t$, $s > 0$, $t > 0$, if $p = 2$.

ACKNOWLEDGEMENT

The author wishes to thank Dr. A. Janssen for valuable remarks concerning the manuscript. These remarks led to a considerable simplification of the proof of Theorem (2.9).

REFERENCES

[1] I. A. Ibragimow and R. Z. Has'minskii, *Statistical estimation (asymptotic theory)*, Springer Verlag, 1981.

[2] A. Janssen, *Unendlich teilbare statistische Experimente*, Habilitationsschrift, Dortmund 1982.

[3] A. N. Kolmogorov, *Wienersche Spiralen und einige andere interessante Kurven im Hilbertschen Raum*, Dokl. Akad. Nauk S.S.S.R. 26 (1940), p. 115-118.

[4] L. LeCam, *On the assumptions used to prove asymptotic normality of maximum likelihood estimates*, Ann. Math. Statistics 41 (1970), p. 802-828.

[5] – *Sur les contraintes imposées par les passages à la limite usuels en statistique*, Bull. I. S. I., Proc. 39th Session, Vienna, Vol. XLV, Book 4 (1973), p. 169-177.

[6] – *Notes on asymptotic methods in statistical decision theory*, Centre de Recherches Mathématiques, Université de Montréal, Montreal 1974.

[7] – *On local and global properties in the theory of asymptotic normality of experiments*, Stoch. Processes relat. Top., Vol. 1 Proc. Summer Res. Inst., Bloomington (1975), p. 13-54.

[8] H. Milbrodt, *Representation of Poisson experiments*, in: *Infinitely Divisible Statistical Experiments*, Lecture Notes in Statistics, Springer, New York 1985.

[9] H. Milbrodt and H. Strasser, *Limits of triangular arrays of experiments*, ibidem.

[10] D. W. Müller, *Contribution to the Discussion of Le Cam*, [5], Bull. I. S. I., Proc. 39th Session, Vienna, Vol. XLV, Book 4 (1973), p. 178-179.

[11] G. Pflug, *A statistically important Gaussian process*, Stochastic Processes and their Application 13 (1982), p. 45-57.

[12] B. L. S. Prakasa Rao, *Estimation of the location of the cusp of a continuous density*, Ann. Math. Statistics 39 (1968), p. 76-87.

[13] H. Strasser, *Statistical Experiments with Independent Increments*, in: *Infinitely Divisible Statistical Experiments*, Lecture Notes in Statistics, Springer, New York 1985.

[14] H. Strasser, *Mathematical Theory of Statistics*, De Gruyter, Berlin 1985.

Universität Bayreuth
Mathematisches Institut
Postfach 3008
D-8580 Bayreuth, BRD

Received on 30. 12. 1982